For the previous block diagram

$$C(s) = G_c(s)\, G_p(s)\, E(s) = G_c G_p E(s) \qquad (1)$$

$$E(s) = R(s) - H(s)\, C(s) = R(s) - H\, C(s) \qquad (2)$$

combining (1 + 2)

$$C(s) = G_c G_p \{ R(s) - H\, C(s) \}$$ → rearrange terms

$$C(s) + G_c G_p H\, C(s) = G_c G_p R(s) \qquad (3)$$

The closed-loop transfer function can now be solved

$$\frac{C(s)}{R(s)} = \frac{G_c G_p}{1 + G_c G_p H} \qquad (4)$$

where $G_c = G_c(s)$ $\quad G_p = G_p(s)$ $\quad H = H(s)$

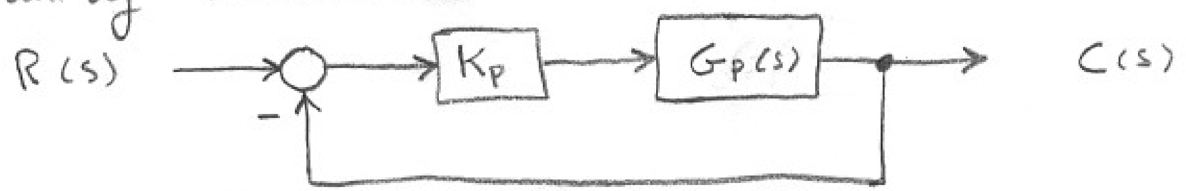
Let's see how we can apply the above closed-loop feed back system to a first order plant

plant ⇒ $G_p(s) = \dfrac{1}{\tau s + 1}$

Proportional Control

For this control system $\quad G_p(s) = K_p = \text{constant}$

Let's also let $H(s) = 1$, this is referred to as "unity feedback"



Now let's compute the closed-loop transfer function

$$C(s) = K_p G_p \{ R(s) - C(s) \}$$

$$\frac{C(s)}{R(s)} = \frac{K_p G_p}{1 + K_p G_p} = \frac{K_p \left(\frac{1}{\tau s + 1}\right)}{1 + K_p \left(\frac{1}{\tau s+1}\right)} \rightarrow$$

$$\frac{C(s)}{R(s)} = \frac{K_p}{\tau s + 1 + K_p}$$

The open-loop system has a pole at $\left(-\frac{1}{\tau}\right)$ while the closed-loop system has a pole at $\left(-\frac{1+K_p}{\tau}\right)$ clearly by providing feed back the system performance can be changed.

Proportional Derivative (PD) Control

Now let's choose a compensator that has the following form

$$G_c(s) = K_p + K_D s = K_p\left(1 + \frac{K_D}{K_p} s\right) = K_D\left(\frac{K_p}{K_D} + s\right)$$

The last expression for $G_c(s)$ has the same basic form as for a lead compensator. Now let's substitute this expression into Eq. 4 on page 4-4 to obtain the closed-loop transfer function of the system

Assume $H(s) = 1$ unity feedback

$$\frac{C(s)}{R(s)} = \frac{G_c G_p}{1 + G_c G_p H} = \frac{K_D\left(\frac{K_P}{K_D} + s\right)\left(\frac{1}{\tau s+1}\right)}{1 + K_D\left(\frac{K_P}{K_D} + s\right)\left(\frac{1}{\tau s+1}\right)}$$

$$\frac{C(s)}{R(s)} = \frac{K_D\left(\frac{K_P}{K_D} + s\right)}{\tau s + 1 + K_D\left(\frac{K_P}{K_D} + s\right)} = \frac{K_D\left(\frac{K_P}{K_D} + s\right)}{s(\tau + K_D) + (1 + K_P)}$$

- The PD control compensator can quicken the response of the uncompensated system or alter the damping or overshoot.

<u>Proportional Integral (PI) Control</u>

Now let's choose a compensator that has the following form: $G_c(s) = K_P + \frac{K_I}{s} = K_P\left(1 + \frac{1}{\frac{K_P}{K_I} s}\right) = \rightarrow$

$$G_c(s) = K_P\left(1 + \frac{1}{T_i s}\right) = \frac{K_P}{s}\left(s + \frac{1}{T_i}\right) \quad \text{where } T_i = \frac{K_P}{K_I}$$

Substituting into Eq. 4 and assuming unity feedback the closed loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{G_c G_p}{1 + H G_c G_p} = \frac{\frac{K_P}{s}\left(s + \frac{1}{T_i}\right)\left(\frac{1}{\tau s + 1}\right)}{1 + \frac{K_P}{s}\left(s + \frac{1}{T_i}\right)\left(\frac{1}{\tau s + 1}\right)}$$

$$\frac{C(s)}{R(s)} = \frac{K_P\left(s + \frac{1}{T_i}\right)}{s(\tau s + 1) + K_P\left(s + \frac{1}{T_i}\right)}$$

Note: The order of the system has increased from 1st to 2nd

The most significant feature of the PI controller is that it can improve steady state error.

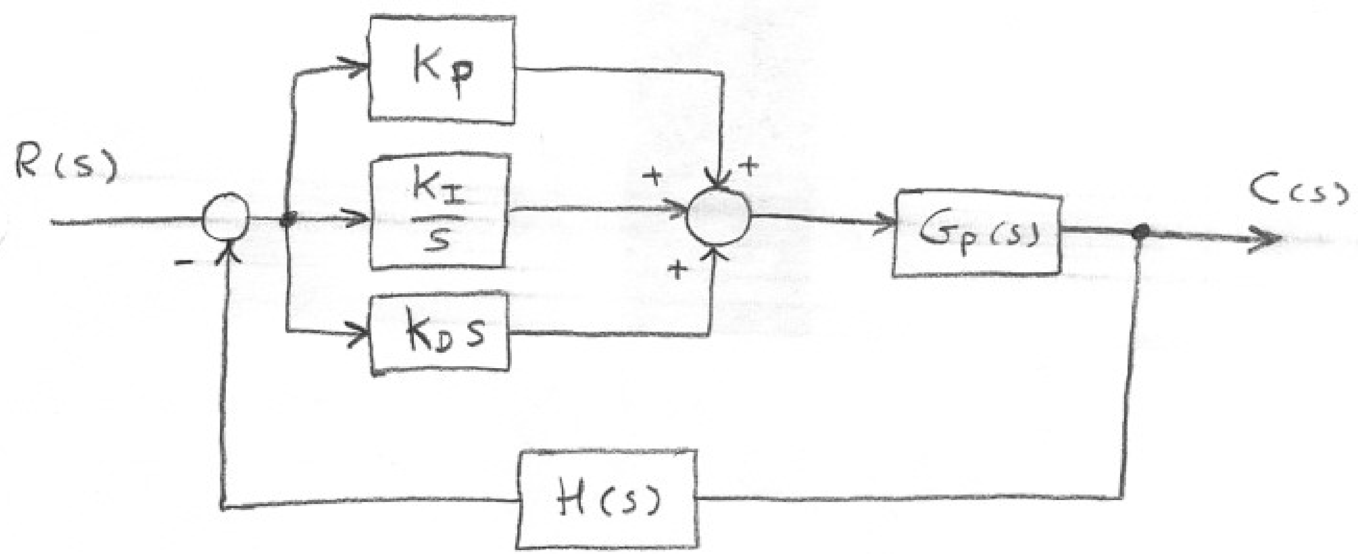
Proportional Integral Derivative (PID) control

Now let's choose a compensator that has the following form:

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s = K_P\left(1 + \frac{1}{sT_i} + T_d s\right)$$

where $T_i = \frac{K_P}{K_I}$ and $T_d = \frac{K_D}{K_P}$

The block diagram of the system is given by:

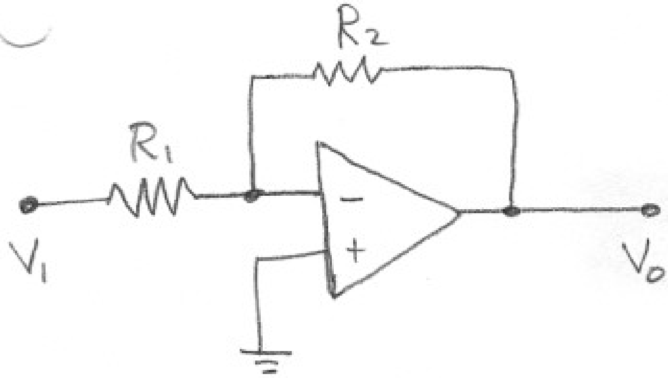


Substituting into Eq. 4 and assuming unity feedback

$$\frac{C(s)}{R(s)} = \frac{G_c G_p}{1 + H G_c G_p} = \frac{\left(K_P + \frac{K_I}{s} + K_D s\right)\left(\frac{1}{\tau s + 1}\right)}{1 + \left(K_P + \frac{K_I}{s} + K_D s\right)\left(\frac{1}{\tau s + 1}\right)}$$

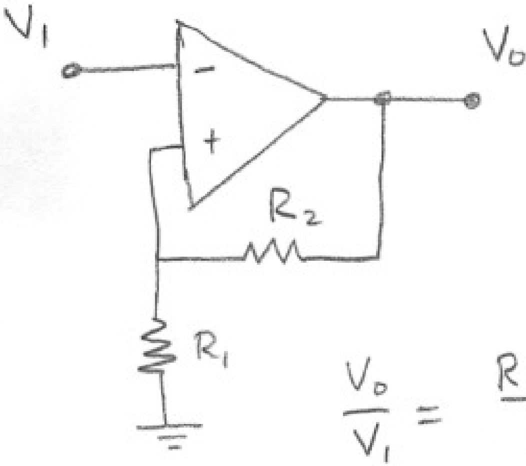
$$\frac{C(s)}{R(s)} = \frac{K_D\left(s^2 + \frac{K_P}{K_D} s + \frac{K_I}{K_D}\right)}{(K_D + \tau)s^2 + (K_P + 1)s + K_I} = \begin{cases}\text{closed} \\ \text{loop} \\ \text{transfer} \\ \text{function}\end{cases}$$

- The gain block, integrator and differentiator can be realized by using electrical circuits



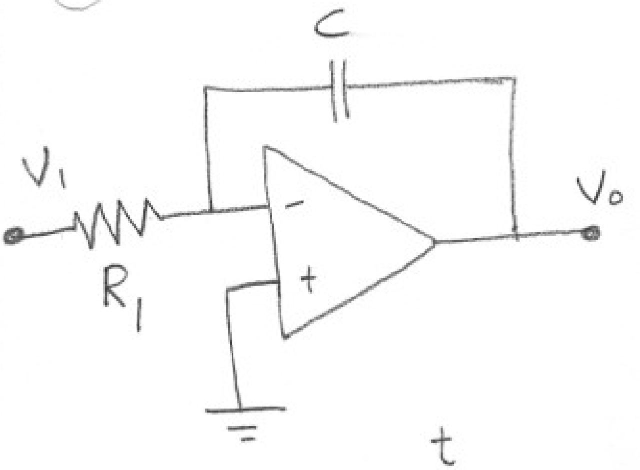
$$V_o = -\frac{R_2}{R_1} V_1$$

Inverting Amplifier



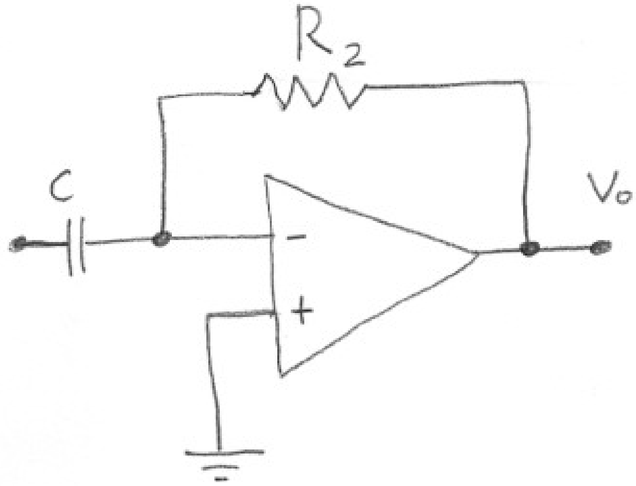
$$\frac{V_o}{V_1} = \frac{R_1+R_2}{R_1}$$

Non- Inverting Amplifier



$$V_o = -\frac{1}{R_1C}\int_0^t V_1\,dt + V_o(0)$$

Integrator



$$V_o = R_2 C \frac{dV_1}{dt}$$

Differentiator